International Journal of Theoretical Physics, Vol. 34, No. 8, 1995

Topological Interference in Nonlinear Bounded Systems

M. Peev[1] and P. Kasperkovitz[1]

Received March 28, 1995

Topological interference is a self-interference of wave packets evolving in quantum wells. A particular manifestation of this self-interference is an effect with no classical counterpart—a symmetric wave packet splitting. Employing reasonable approximations, we give a semianalytical interpretation of this effect.

1. INTRODUCTION

In the present paper we discuss a quantum effect that arises in the course of the Schrödinger evolution of a well-defined family of localized initial states (typically wave packets) in one-dimensional bounded anharmonic (nonlinear) systems. As a consequence of a specific quantum self-interference of the evolving state, which we refer to as topological interference, it splits into a linear combination of disjointly and symmetrically localized wave packets. The splitting occurs only in particular, relatively short time intervals, the type of the splitting being different for each of these intervals. The evolution time which passes until such an interval is reached is much longer than the interval itself. The studied effect demonstrates the essential long-time deviations of classical and quantum evolutions, even in cases when these evolutions are asymptotically close in the short-time scale.

The material is organized as follows. In Section 2 we briefly introduce the Q-phase-space representation of quantum mechanics (QM), which is a particularly natural language for presenting topological interference. This interference and the subsequent splitting effect are introduced in Section 3. In Section 4 we present an analytical result. It allows us, in the framework of an adequate spectral approximation, to interpret the action of the propagator

[1] Institut für Theoretische Physik, Technische Universität Wien, A-1040 Vienna, Austria.

at specific time instants as a sum of symmetric "pseudotranslations." Thus we arrive at a theoretical explanation of the wave packet splitting effect. In the concluding Section 5 we discuss briefly the general conditions for occurrence of the splitting effect.

2. Q-PHASE-SPACE FORMALISM

In the Q-phase-space representation of quantum mechanics (a transcription of the usual Hilbert space version) the noncommutative ring $\mathcal{A}$ of operators defined on $\mathcal{H}$ (the quantum Hilbert space) is mapped onto a noncommutative ring $\mathcal{B}$ of functions defined on the classical phase space (p, q), so that $\mathcal{A} \cong \mathcal{B}$. The mapping $\mathcal{A} \leftrightarrow \mathcal{B}$ is defined using the canonical coherent states $|p, q\rangle$, $p, q \in \mathbf{R}$,

$$\mathcal{A} \ni \hat{A} \leftrightarrow A(p, q) = \langle p, q|\hat{A}|p, q\rangle \in \mathcal{B} \tag{1}$$

The function $A(p, q)$ is known as the Q-phase-space representative of the operator $\hat{A}$.

The Q-representatives $W_H(p, q)$ of all density operators $\hat{\rho}$ (corresponding to the family of all pure and mixed states in $\mathcal{H}$), are called Husimi functions. The Husimi functions are real and nonnegative functions on the classical phase space (p, q); their integral over this space is equal to one. Therefore they are joint phase-space probability distributions. These can be interpreted as the distributions corresponding to the QM states with respect to a well-defined unsharp joint measurement of the position and momentum observables (Busch *et al.*, 1991). The unsharpness is best demonstrated by taking the marginals of $W_H(p, q)$,

$$\overline{W}(p) = \int_{-\infty}^{+\infty} W_H(p, q)\, dq; \qquad \overline{W}(q) = \int_{-\infty}^{+\infty} W_H(p, q)\, dp \tag{2}$$

The probability distributions $\overline{W}(s)$, $s = p, q$, are related to the usual sharp QM probabilities $W(s)$, $s = p, q$ $[W(s) = \langle s|\hat{\rho}|s\rangle$; for pure states $W(s) = \|\psi(s)\|^2]$, through the smoothing convolution

$$\overline{W}(s) = \int_{-\infty}^{+\infty} W(s_0) \exp\left\{-\frac{(s - s_0)^2}{2\hbar}\right\} ds_0, \qquad s = p, q \tag{3}$$

On all figures below, the Husimi functions of QM states are presented. They are supplemented by the sharp position distributions $W(q)$ (dashed line) and the respective convoluted distributions $\overline{W}(q)$ (solid line).

3. THE EFFECT

3.1. The Hamiltonian

The splitting effect occurs in one-dimensional one-particle quantum systems, characterized by a smooth potential $V(x)$ $[V(x \to \pm\infty) \to +\infty]$ that has a single minimum. Typically these systems are anharmonic oscillators. A family of Hamiltonians of this type is, e.g.,

$$\hat{H} = \frac{\hat{p}^2}{2\mu} + c_1\hat{q}^2 + c_2(\hat{q}^2)^\gamma, \qquad 1 < \gamma \leq \infty, \quad c_1 \geq 0, \quad c_2 > 0. \tag{4}$$

This family is bounded by two analytically solvable models: the harmonic oscillator ($c_2 = 0$) and the square well ($c_1 = 0$, $\gamma \to \infty$). The harmonic oscillator is excluded from the family as it is an exceptional case for which the effect does not hold. (It is well known that the quantum propagation generated by quadratic Hamiltonians is essentially classical.)

3.2. The Initial Condition

Sufficient conditions on the initial state $|\psi_{t=0}\rangle$ needed for a clear observation of the considered effect are the following:

(i) The initial state has to be *sufficiently localized in phase space*: The values of Husimi function of the initial state outside a finite region should be sufficiently small. Stated otherwise:

$$\exists a: \quad D(W(p)) < a \quad \text{and} \quad D(W(q)) < a \tag{5}$$

where a is a sufficiently small constant; $D(W(\circ))$ is the standard deviation of the probability distribution $W(\circ)$; and $W(s)$, $s = p, q$ is defined as above.

(ii) The initial state has to be "*moderately localized in energy*": Let the expansion of $|\psi_{t=0}\rangle$ in eigenvectors $|\psi_n\rangle$ of the Hamiltonian $\hat{H}$ be

$$|\psi(s)_{t=0}\rangle = \sum_n a_n|\psi_n\rangle$$

$$a_n \approx 0 \quad \text{iff} \quad n < n_1, n > n_2 \tag{6}$$

The present condition requires that

$$\exists b, c: \quad b < n_2 - n_1 < c \tag{7}$$

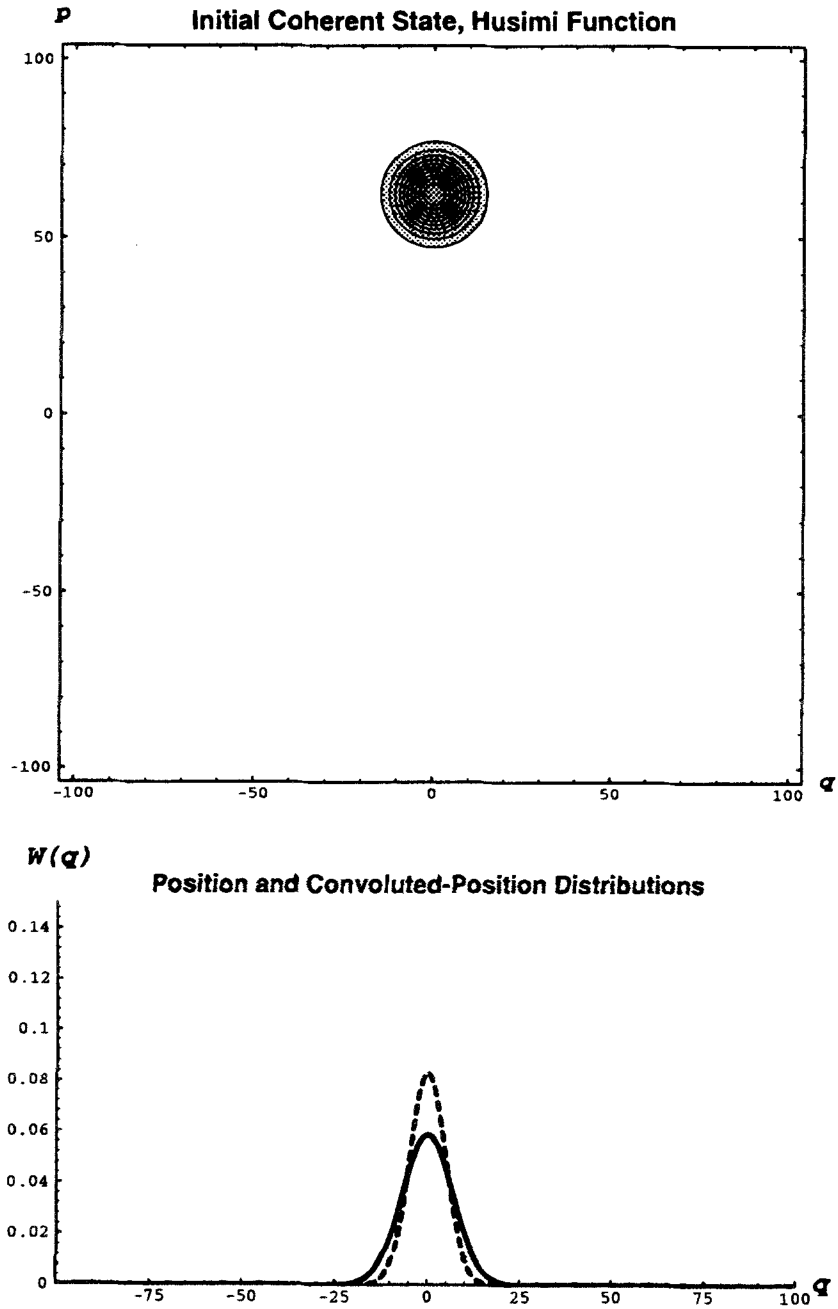
where b is not too small and c not too big. This implies that the initial state should essentially lie in a finite-dimensional eigenspace of $\hat{H}$ of "moderate" (neither too small nor too big) dimension $n_2 - n_1$.

A typical state that complies with these conditions is a coherent state, the average energy of which is neither too high nor too low. (The first condition is automatically fulfilled for any coherent state.)

3.3. The Time Evolution

Several characteristic stages in the evolution of one such coherent state (Fig. 1, initial state) are presented in Figs. 1–5. [The time units are relative. The results are obtained numerically. We have chosen a potential $V(x)$ with $c_1 = 0.05$, $c_2 = 0.00075$, $\gamma = 6$.] The mentioned stages are:

- *Semiclassical evolution* (Fig. 2, $T = 0.15$). Both the sharp (exact) and the smeared q (and p) probability distributions evolve classically. The quantum evolution of the Husimi function is very close to a Classical evolution. This stage has been extensively investigated (e.g., Heller, 1977; Littlejohn, 1986). The typical feature (of both the classical and the semiclassical) evolution is the stretching of the phase-space distribution along the classical trajectories, which arises as a consequence of the anharmonicity.
- *Quasiclassical evolution* (Fig. 3, $T = 0.30$). There is no qualitative change in the Husimi function evolution. Its stretching continues. The sharp probability distribution gets a reflection interference pattern. The reflection interference can be qualitatively described as follows. It occurs when parts of the wave packet meet each other while moving in opposite directions. This interference is obviously stronger when the packet is more stretched. As the wave length of the interference pattern is inversely proportional to the relative speed of meeting parts, the reflection interference is in general a short-wave one (provided that the energy of the initial wave packet is not too low). For this reason it is not "seen" by an unsharp observable and does not affect essentially the appearance of the Husimi function.
- *Beginning of topological interference* (Fig. 4, $T = 0.45$). The head and tail of the spreading wave packet meet. A new interference starts. The Husimi function forms a closed loop in phase space. (This is the reason for the name we use—topological interference.) In configuration space quick parts of the wave packet start overriding its slow parts. Now the relative speed of the meeting parts is low. The interference is a long-wave one. Therefore the interference pattern is unavoidably detected even with unsharp observables and an interference pattern is also seen on the Husimi function. This is the moment when the classical and quantum evolutions essentially depart from each other. (Note that it is the Q-phase-space representation that allows us clearly to distinguish this stage from the previous one.)
- *Wave packet splitting effect* (Fig. 5, $T = 1.667$). At certain well-defined moments of time ($\ldots, T_m = T_1/m, \ldots, T_4 = T_1/4, T_3 = T_1/3, T_2 = T_1/2, T_1, T_0 = 2\,T_1$), the evolving wave packet splits into $\ldots, m, \ldots, 4, 3, 2, 1$ similar pieces. In the last of these moments ($t = T_0$) the initial and the evolving state almost coincide. (A + $T = T_1$ there is also only one wave packet, similar to the initial one, but it is with inverse momentum.) The splitting persists for short time intervals, while the separate parts evolve



p
Initial Coherent State, Husimi Function
100
50
0
-50
-100
-100
-50
0
50
100
q
W(q)
Position and Convoluted-Position Distributions
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
-75
-50
-25
0
25
50
75
100
q

Fig. 1.

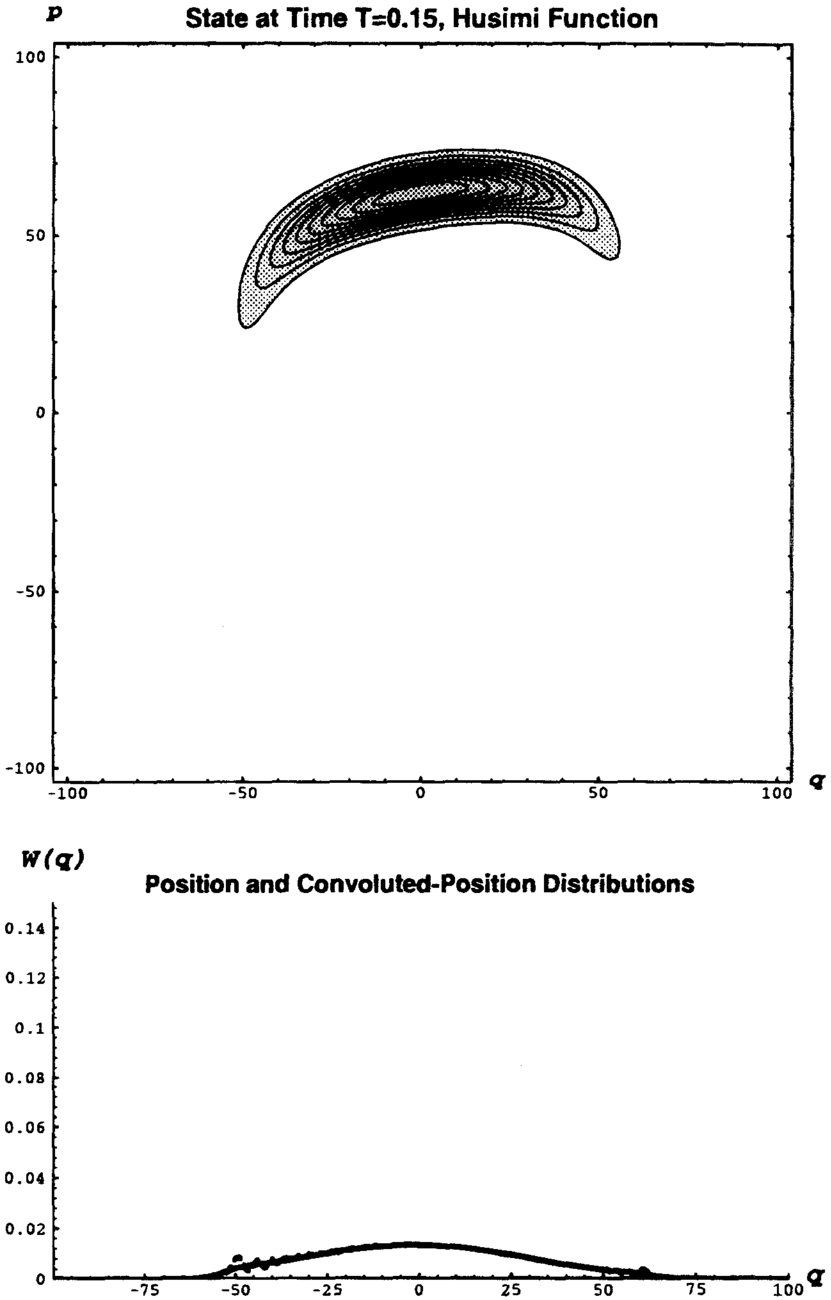

p
State at Time T=0.15, Husimi Function
100
50
0
-50
-100
-100
-50
0
50
100
q
W(q)
Position and Convoluted-Position Distributions
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
-75
-50
-25
0
25
50
75
100
q

Fig. 2.

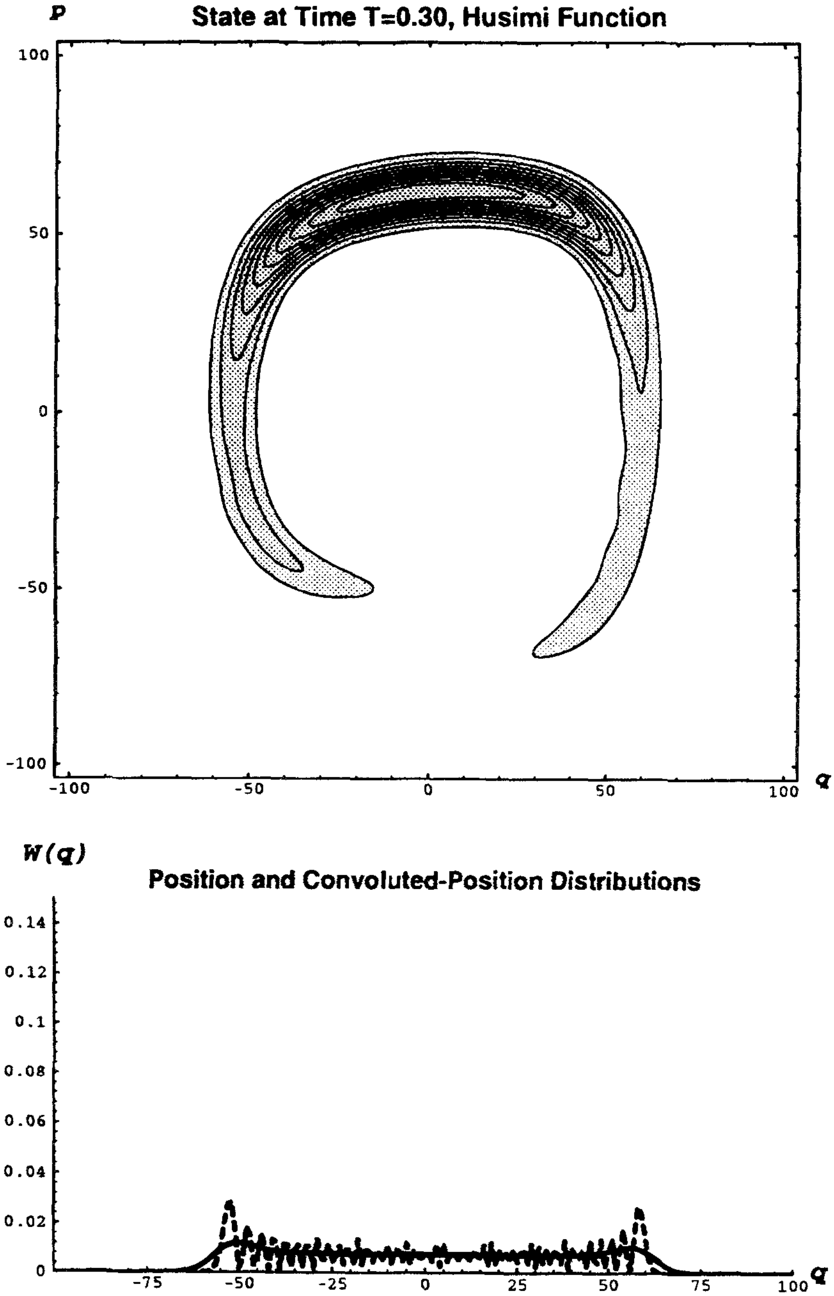
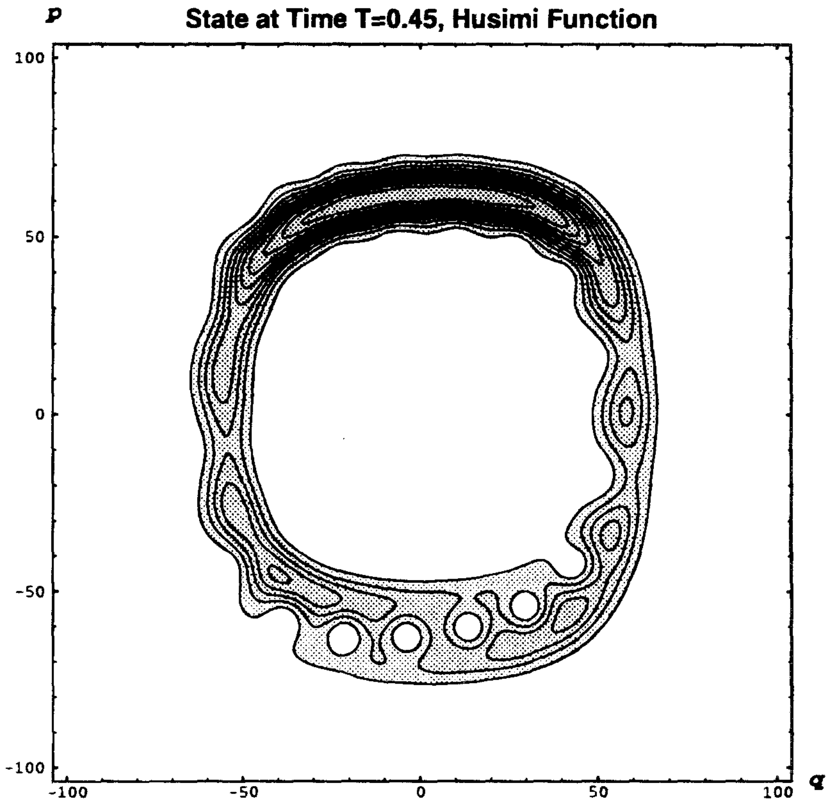

p
State at Time T=0.30, Husimi Function
100
50
0
-50
-100
-100
-50
0
50
100
q
W(q)
Position and Convoluted-Position Distributions
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
-75
-50
-25
0
25
50
75
100
q

Fig. 3.


State at Time T=0.45, Husimi Function
p
100
50
0
-50
-100
-100
-50
0
50
100
q

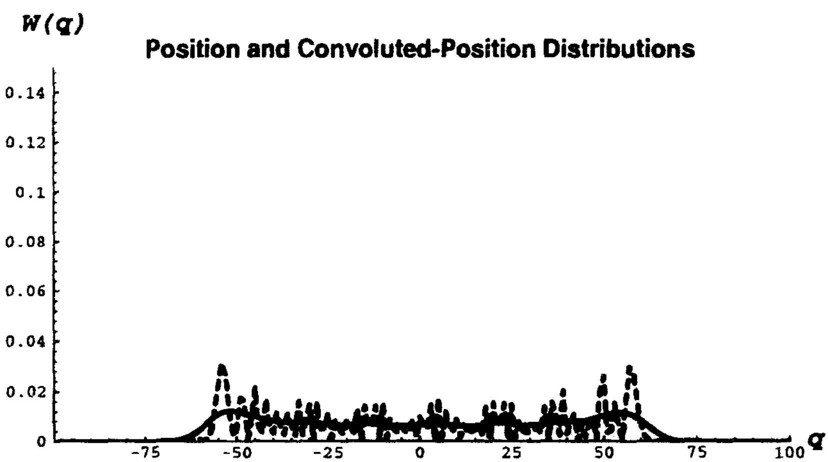

W(q)
Position and Convoluted-Position Distributions
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
-75
-50
-25
0
25
50
75
100
q

Fig. 4.

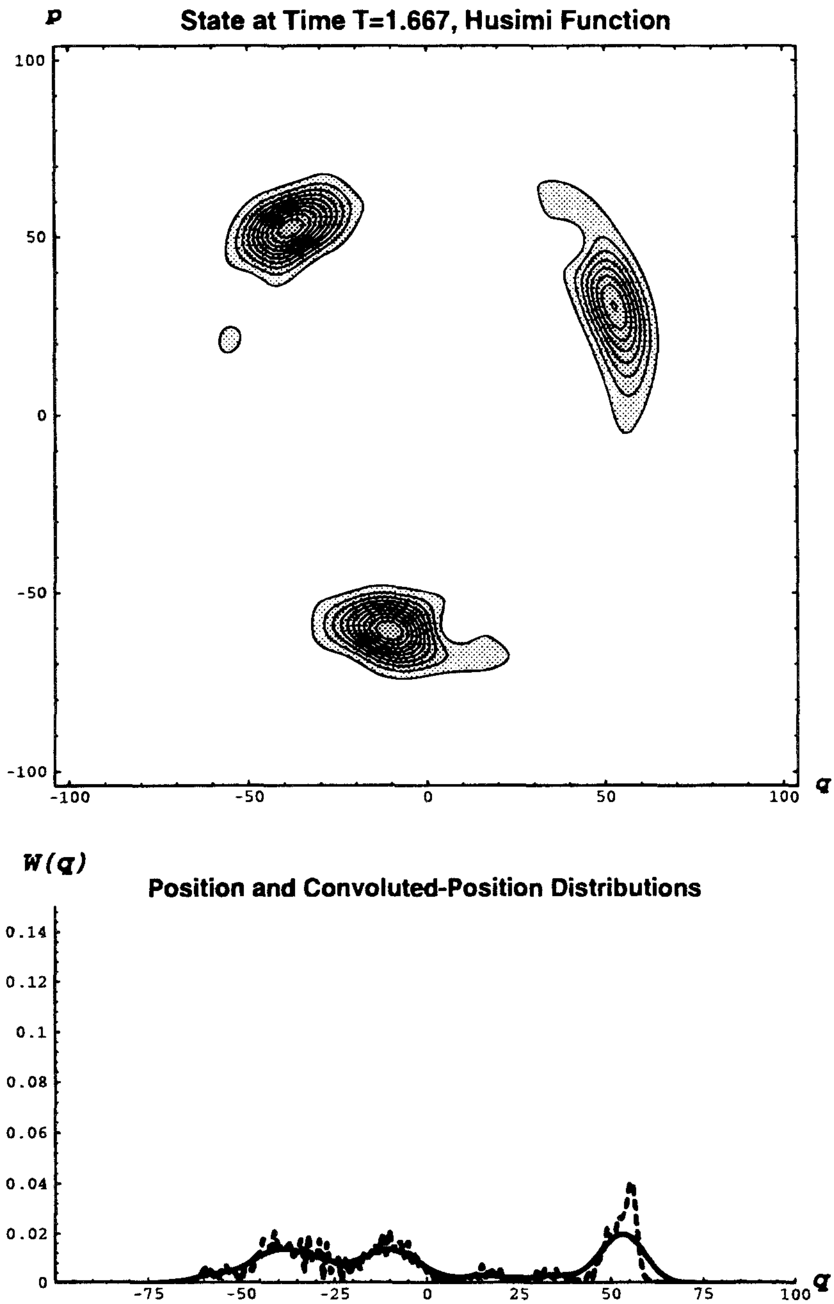

State at Time T=1.667, Husimi Function
p
q
Position and Convoluted-Position Distributions
W(q)

Fig. 5.

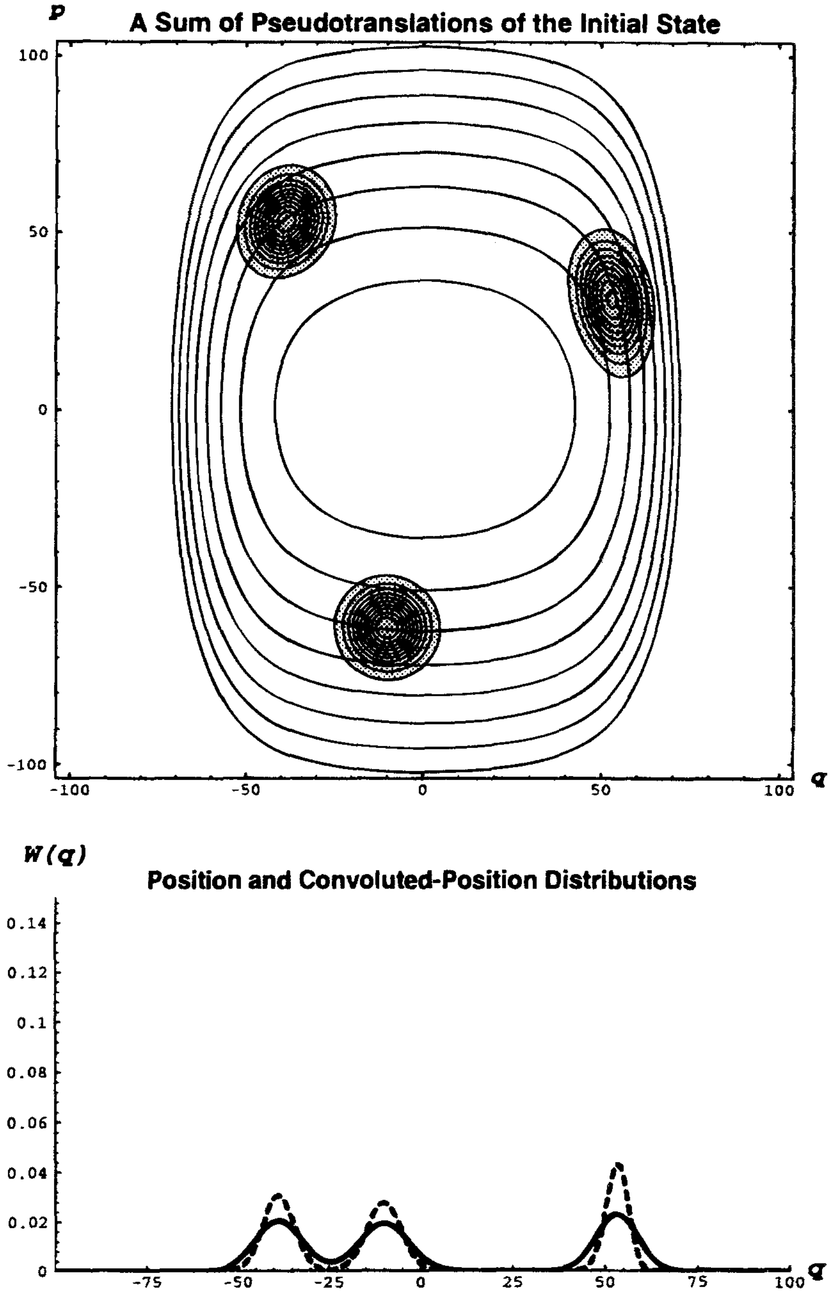

p
A Sum of Pseudotranslations of the Initial State
100
50
0
-50
-100
-100
-50
0
50
100
q
W(q)
Position and Convoluted-Position Distributions
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
-75
-50
-25
0
25
50
75
100
q

Fig. 6.

semiclassically and stretch. Each such intervals ends when the disjoint parts start interfering with each other.

The first three stages of this quantum evolution are qualitatively well understood. This holds also for $t = T_0$. (The initial state belongs to an essentially finite-dimensional eigenspace. Therefore, as long as there always exists a sufficiently good rational approximation of the spectrum, the Schrödinger evolution is essentially recurrent.) To our knowledge, however, the wave packet splitting effect has not yet been studied.

4. A SEMIANALYTICAL INTERPRETATION

To proceed further we make use of the following, easy to prove, equation:

$$\exp\left\{-in^2\frac{\pi}{m}\right\} = \sum_{k=1}^{m} \alpha(m)_k \exp\left\{-i2\pi\left(\frac{kn}{m} + \frac{n}{2}\right)\right\}, \qquad \forall m, n = 1, 2, \ldots \tag{8}$$

$$\alpha(m)_k = \frac{\alpha(m)}{m} \exp\left\{-i\frac{\pi}{m}(k^2 + km)\right\}; \qquad \alpha(m) = \sum_{s=1}^{m} \exp\left\{i\frac{\pi}{m}(s^2 - sm)\right\} \tag{9}$$

$$\|\alpha(m)_k\| = \frac{1}{m}, \qquad \alpha(m)_k = \alpha(m)_{m-k}, \qquad k = 1, \ldots, m \tag{10}$$

Next let us consider a quadratic approximation of the spectrum of $\hat{H}$:

$$\tilde{E}_n = a_1 n + a_2 n^2 \tag{11}$$

where the coefficients a_1 and a_2 are chosen so that $\tilde{E}_n$ is the best mean square fit to the true eigenvalues E_n in the spectral region determined by the boundaries n_1 and n_2 [see condition (ii) above]. The general properties of spectra of Hamiltonians of the discussed type guarantee that in the respective eigenspace this is a reasonable approximation provided that its dimension is not too big. Therefore the action of the propagator onto an initial state complying with condition (ii) can be adequately approximated as follows:

$$\begin{aligned}\hat{U}(t)|\psi_{t=0}\rangle &= \left[\sum_n \exp\left\{-\frac{iE_n t}{\hbar}\right\}|\psi_n\rangle\langle\psi_n|\right]|\psi_{t=0}\rangle \\ &\approx \left[\sum_n \exp\left\{-\frac{i\tilde{E}_n t}{\hbar}\right\}|\psi_n\rangle\langle\psi_n|\right]|\psi_{t=0}\rangle\end{aligned} \tag{12}$$

From equations (8)–(12) we conclude that at special moments of time $T = T_m = \frac{\pi}{m}$

$$\hat{U}(T_m)|\psi_{t=0}\rangle \approx \sum_{k=1}^{m} \alpha(m)_k \hat{\mathcal{T}}\left(\frac{k}{m} + \frac{1}{2} + \tau\right)|\psi_{t=0}\rangle \tag{13}$$

$$k = 1, \ldots, m, \quad m = 1, 2, \ldots$$

$$\hat{\mathcal{T}}(\beta) := \hat{\mathcal{T}}(\beta)|\psi_n\rangle = \exp\{-i2\pi\beta n\}|\psi_n\rangle, \forall n; \qquad T = \frac{a_2}{\hbar}t, \quad \tau = \frac{a_1}{a_2}T \tag{14}$$

Here T is time measured in scaled time units. (Note that in the figures time has been rescaled by a factor $\frac{5}{\pi}$.) In the exceptional cases of the harmonic oscillator and the square well the operators $\hat{\mathcal{T}}(\beta)$ are readily seen to be "pseudotranslations": they translate the phase-space representative of any state along the classical orbits of the system. This interpretation cannot be directly transferred to more general cases, as then the phase-space action of the operator $\hat{\mathcal{T}}$ cannot be derived analytically. There is, however, good numerical evidence that for any anharmonic oscillator $\hat{\mathcal{T}}(\beta)\,|\,\psi(s)_{t=0}\rangle$ is indeed a pseudotranslation, supplemented by a certain (weak) deformation. Figure 6 presents the Husimi function of the state on the RHS of equation (13). A comparison with Fig. 5, which is the Husimi function of the state on the LHS of equation (13) ($T \approx T_3$, $m = 3$) shows that the qualitative similarity of the two figures is satisfactory. On top of the Husimi function in Fig. 5 we have plotted the classical phase portrait of the Hamiltonian at hand. It is clearly seen that the operator $\hat{\mathcal{T}}$ performs pseudotranslations of the initial state.

5. CONCLUSION

In the preceding discussion we did not comment on all conditions imposed on the initial state. First we note that condition (i) is merely required for a clearer observation of the splitting. (If the initial state were not localized, the pseudotranslated ones could overlap.) In the previous section it became rather obvious why $n_2 - n_1$ should not be big [see condition (ii)]. If $n_2 - n_1$ were too small and condition (i) is fulfilled, then it can be shown that the pseudotranslated states would strongly overlap and would tend to coincide.

For more general bounded one-dimensional Hamiltonians than the one introduced in Section 3.1 the effect would still be recognizable, but probably in combination with other effects (e.g., tunneling for a double well). The reason is that equation (11) is not always a good spectral approximation even in a low-dimensional subspace.

Finally, the effect could be observed in higher dimensional bounded systems, but only in certain special integrable cases.

ACKNOWLEDGMENT

This work was supported by the Austrian Science Foundation (FWF) under project M0169-PHY.

REFERENCES

Busch, P., Lahti, P. J., and Mittelstaedt, P. (1991). *The Quantum Theory of Measurement*, Springer-Verlag, Berlin.

Heller, E. J. (1977). *Journal of Chemical Physics*, **67**, 3339–3351.

Littlejohn, R. G. (1986). *Physics Reports*, **138**, 193–291.